Optimization and Distance Function Procedures in Aberration Criteria of Efficient Fractional Factorial Design $OA(n, 2^1 3^1 4^1)$

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Abstract — An efficient orthogonal array was constructed with near balance and near the orthogonal property for the lowest common multiples of runs, using the balance coefficient criteria for determining near balance and J_2 optimality criteria for orthogonal properties. The optimization and distance function forms of balance coefficient criteria were used for the classification of the designs. The Minimum Moment Aberration (MMA) and Minimum Aberration Projection (MAP) are compared using the optimization and distance function to determine the near balance criteria. The result indicated that, the MMA and MAP criteria was efficient using the optimization procedure of the balance coefficient.

Keywords: Balance Coefficient, Distance Function, Minimum Moment Aberration, Minimum Aberration Projection, Orthogonal, Optimization.

1 Introduction

Factorial designs have broad applications in agricultural, engineering, and scientific studies. In constructing and studying properties of factorial designs, traditional design theory treats all factors as nominal. However, this is not appropriate for experiments that involve quantitative factors. For designs with quantitative factors, a level permutation of one or more factors in a design matrix could result in different geometric structures, and, thus, different design properties. Two or three – levels factorial experiments are mostly used in the design of experimental research. In many situations, factors with more than two-three levels are desirable, when the factors are either qualitative or quantitative. As a result, designs with mixed-level factors have been used more often to design experiments in modern industrial and agricultural trials, especially when only limited resources are allowed. Full factorial designs are test matrices that contain all possible combinations of the levels of the factors. For example, if factor A has a level, factor B has b levels and factor C has c levels, then the full factorial design will contain *abc* combinations. The shorthand notation for this design is ($4^1 3^1 2^1$), which displays the levels of the factor as the base numbers and the number of the factor as the exponent.

Of the desirable properties of factorial experiments are balance and orthogonal. Balance requires a level of factors replicated the same number of times as any other level of this factor in an experiment. Orthogonal designs are pairwise linearly independent, useful for assessing factor significance. As the factor levels increase, the number of runs increases, and maintaining the balance property requires too many runs in some situations.

For example, consider a design with four factors, one with three levels, one with five levels, one with five levels, and the last with two levels. To generate a balanced design, at least 150 runs are needed. Suppose an experimenter only has resources for 50 tests, and the test objective is screening. Then, a mechanism for creating mixed-level designs that are capable of meeting desirable resources is required.

Minimum aberration has been widely recognized as a useful criterion for selecting regular fractional factorials. Recent work on minimum aberration designs includes Chen and Ye (2004), Tang and Wu (1996), Chen and Hedayat (1996), and Cheng et al. (1999). Minimum aberration mixed-level designs are also balanced, Cheng et al, (1999), Deng and Tang (2002), Mukerjee and Wu (2001), Xu and Wu (2001). In situations where we have little or no knowledge about the effects that are potentially important, it is appropriate to select designs using the minimum aberration criterion [Fries and Hunter (1980)]. Wu and Hamada (2000) contain tables of many known minimum aberration designs. Minimum aberration designs enjoy some attractive robust properties [Cheng, Steinberg, and Sun (1999) and Tang and Deng (1999)]. Much work has been done on the construction of minimum aberration designs. For details, we refer to Franklin (1984), Chen and Wu (1991), Chen (1992), Chen and Hedayat (1996), Tang and Wu (1996), Suen, Chen and Wu (1997) and many others. Sitter, Chen, and Feder (1997), Chen and Cheng (1999) and Cheng and Wu (2002) developed aberration criteria for blocked fractional factorials. For unbalanced mixed-level fractional factorial designs, the degree of balance was evaluated using a balance coefficient (Guo (2003)). As an extension of two level fractional factorial designs, Franklin (1984) and Suen, Chen and Wu (1997) discuss the construction of multi-level minimum aberration designs. Xu and Wu (2001) proposed a generalized minimum aberration for mixed –level fractional factorial designs. Wang and Wu (1992) and Ankenman (1999) used minimum aberration designs in two-level and four – level designs. Murkerjee and Wu (2001) developed minimum aberration designs for mixed-level fractional factorial designs involving factors with two or three distinct levels. The objective of this paper is to compare the two forms of balance coefficient in a fractional factorial design using Minimum Moments Aberration and Minimum Aberration Projection at various runs sizes.

2.0 Formulation of Balance Coefficient – Form I

In form I, the motivation behind the definition of the balance coefficient is a simple optimization problem. The balance coefficient of design matrices will be derived from this optimization problem can be formulated as,

$$\text{Max} \qquad G = \prod_{K=1}^{K} X_K \tag{1}$$

$$\text{Subject to} \qquad \sum_{k=1}^{k} X_K = C, \tag{2}$$

Where C is a constant.

The balance coefficient for design matrix k, $F(k)$, is defined as the combination of the balanced coefficient of each column, F_j,

$$F(k) = \sum_{j=1}^{m} w_j F_J = \sum_{j=1}^{m} \left(\prod_{i=1}^{l_j} l_{ij} \right) w_j,$$

w_j are the weights for the corresponding columns.

This balance coefficient depends on the runs. To avoid this situation, a standardized balance coefficient is defined by using a standardized number of levels. The balanced coefficient is standardized when the number of levels is standardized. The notations f_{ij} is used instead of l_{ij}. In this for a specific column and for a design matrix.

3.0 Formulation of Balance Coefficient – Form II

In form II, the definition of balance coefficient employs the

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concept of the distance function. Consider a distance function-

$$H_J = \sum_{i=1}^{lj} (l_{ij} - T)^2,$$

where $T = n/l_j$, is a fixed value.

The balance coefficient under this definition becomes.

$$H_j = \sum_{i=1}^{l_j} (l_{ij} - \frac{n}{l_j})^2$$

and

$$H = \sum_{j=1}^{m} H_j = \sum_{j=1}^{m} w_j \sum_{i=1}^{l_j} (l_{ij} - \frac{n}{l_j})^2$$

If f_{ij} are used instead of l_{ij}, then standardized H_j and H can be given by

$$\hat{H}_j = \sum_{i=1}^{l_j} (f_{ij} - \frac{1}{l_j})^2, \text{ and}$$

$$\hat{H} = \sum_{j=1}^{m} w_j \, H_j = \sum_{j=1}^{m} w_j \sum_{i=1}^{\hat{l}_j} (f_{ij} - \frac{1}{l_j})^2$$

4.0 Minimum Moment Aberration Criterion

The *MGA*, *MG₂A*, and *GMA* criteria all require contrast coefficients of factors. Xu (2003) developed a Minimum Moment Aberration criterion (MMA), which does not need contrast coefficients. For a design matrix *d*, d_{ij} be the elements of i^{th} row and j^{th} column. The coincidence between two elements d_{ij} and d_{lj} is defined by $\delta(d_{ij}, d_{lj})$, where $\delta(d_{ij}, d_{lj})$ = 1 if $d_{ij} = d_{lj}$ and 0 otherwise. The value of $\sum_{j=1}^{m} \delta(d_{ij}, d_{lj})$ measures the coincidence between i^{th} and j^{th} rows of *d*. The k^{th} power moment is defined by Xu (2003) as

$$K_k(d) = [n(n-1)/2]^{-1} \sum_{1 \le i \le l \le n} \left[\sum_{j=1}^{m} (d_{ij}, d_{lj}) \right]^k .$$

For two designs d_1 and d_2, d_1 is said to have less moment aberration than d_2 if there exists an r such that $K_r(d_1) < K_r(d_2)$ and $K_t(d_1) = K_t(d_2)$ for all $t=1, \ldots, r-1$. Therefore, d_1 is said to have minimum moment aberration if there is no other design with less moment aberration than d_1.

5.0 Table 1: Designs using Minimum Moment Aberration Criteria (MMA) in $OA(n, 2^1 3^1 4^1)$

	Minimum Moment Aberration Criteria		
Runs	**Designs**	$(K_1(d), K_2(d), K_3(d), K_4(d))$	
6	Distance Function	(1.267, 2.6, 5.667, 13)	d_2
	Optimization Procedure	(0.8, 1.6, 3.6, 8.8)	
7	Distance Function	(1.429, 3.667, 9.476, 26.43)	
	Optimization Procedure	(0.857, 1.714, 4.809, 10.286)	d_2
8	Distance Function	(1.643, 4.143, 11.286, 32)	
	Optimization Procedure	(0.821, 1.679, 3.964, 10.607)	d_2
9	Distance Function	(1.75, 4.611, 12.778, 36.611)	
	Optimization Procedure	(0.88, 1.94, 4.88, 13.28)	d_2
10	Distance Function	(1.889, 4.467, 11.622, 31.933)	
	Optimization Procedure	(0.88, 1.82, 3.67, 10.62)	d_2
11	Distance Function	(1.727, 4.2, 10.545, 28.2)	
	Optimization Procedure	(0.872, 2.036, 4.873, 12.509)	d_2
12	Distance Function	(1.697, 4.181, 11.060, 30.879)	
	Optimization Procedure	(0.909, 1.939, 4.727, 12.485)	d_2
13	Distance Function	(1.513, 3.487, 8.897, 24.103)	
	Optimization Procedure	(1.025, 2.077, 5.103, 13.462)	d_2
14	Distance Function	(2.494, 2.978, 7.132, 17.703)	
	Optimization Procedure	(1, 2.418, 10.099, 16.923)	d_2
15	Distance Function	(1.352, 2.971, 7.038, 18.371)	
	Optimization Procedure	(0.96, 2.37, 6.314, 17.457)	d_2
16	Distance Function	(1.267, 2.75, 7.05, 19)	
	Optimization	(1.1, 2.608, 6.55, 17.483)	d_2

	Procedure		
17	Distance Function	(1.235, 6.169, 9.279, 26.779)	
	Optimization Procedure	(0.98, 4.93, 5.78, 15.93)	d_2
18	Distance Function	(1.261, 3.366, 10.189, 26.634)	
	Optimization Procedure	(0.987, 2.229, 5.693, 15.562)	d_2

The minimum aberration criteria for two selected designs using form I (Maximum) and form II (Minimum) method of balance coefficient, for $6 \le n \le 18$.
The observation shows that at $n = i, where i = 6,\dots,18$

$$K_1(\mathrm{d}_2) < K_1(\mathrm{d}_1);$$

This indicated that in all the runs mentioned, $K_1(\mathrm{d}_2)$ has a lesser aberration than $K_1(\mathrm{d}_1)$ i.e. the design d_2 is a better fractional factorial of all possible designs in the runs considers.

6.0 Moment Aberration Projection Criterion

Xu and Deng (2005) proposed a criterion called the Moment Aberration Projection (*MAP*) to address the drawback that kth power moment is not corresponding to k-factor interactions. *MAP* uses the coincidence matrix for all factor projections.

For a given k (I ≤ k ≤ m), there are $\binom{m}{k}$ *k*-factor projections.

The frequency distribution f K_k-values of these projections is called the *k*-dimensional K-values distribution and is denoted by $F_k(d)$. For two designs d_1 and d_2, suppose that r is the smallest integer such that the r-dimensional K-value distributions are different, that is, $F_r(d_1) \neq F_r(d_2)$. Hence, d_1 is said to have less *MAP* than d_2 if $F_r(d_1) < F_r(d_2)$.

Moreover, the criterion *MAP* was developed for two-level non-regular designs and it also can be used in multi-level and mixed-level designs.

7.0 Table 2: Designs using Minimum Aberration Projection (MAP) in $OA(n, 2^1 3^1 4^1)$

Runs	Distance Function (Minimum)			Optimization Procedure (Maximum)		
6	A = 10	AB = 28	ABC = 85	A = 6	AB = 13	ABC = 54
	B = 6	AC = 19		B = 3	AC = 15	
	C = 3	BC = 13		C = 3	BC = 8	
7	A = 15	AB = 45	ABC = 199	A = 9	AB = 23	ABC = 101
	B = 10	AC = 33		B = 6	AC = 18	
	C = 6	BC = 28		C = 3	BC = 13	
8	A = 21	AB = 66	ABC = 316	A = 12	AB = 29	ABC = 111
	B = 15	AC = 51		B = 6	AC = 24	
	C = 10	BC = 45		C = 4	BC = 17	
9	A = 28	AB = 181	ABC = 460	A = 16	AB = 39	ABC = 176
	B = 21	AC = 73		B = 9	AC = 33	
	C = 15	BC = 61		C = 7	BC = 30	
10	A = 36	AB = 120	ABC = 523	A = 20	AB = 52	ABC = 165
	B = 28	AC = 87		B = 12	AC = 40	
	C = 21	BC = 79		C = 8	BC = 30	
11	A = 37	AB = 129	ABC = 580	A = 25	AB = 70	ABC = 268
	B = 36	AC = 87		B = 17	AC = 55	
	C = 28	BC = 106		C = 10	BC = 39	
12	A = 39	AB = 119	ABC = 730	A = 30	AB = 80	ABC = 312
	B = 37	AC = 119		B = 18	AC = 66	
	C = 36	BC = 145		C = 12	BC = 46	
13	A = 42	AB = 123	ABC = 694	A = 36	AB = 97	ABC = 398
	B = 39	AC = 151		B = 23	AC = 83	
	C = 37	BC = 126		C = 15	BC = 71	

14	A = 46	AB = 156	ABC = 649	A = 42	AB = 103	ABC = 919
	B = 42	AC = 131		B = 26	AC = 102	
	C = 39	BC = 119		C = 20	BC = 78	
15	A = 51	AB = 173	ABC = 739	A = 49	AB = 129	ABC = 663
	B = 46	AC = 137		B = 31	AC = 113	
	C = 42	BC = 132		C = 25	BC = 82	
16	A = 57	AB = 158	ABC = 846	A = 56	AB = 172	ABC = 786
	B = 47	AC = 159		B = 39	AC = 153	
	C = 47	BC = 169		C = 37	BC = 126	
17	A = 64	AB = 238	ABC = 1262	A = 64	AB = 180	ABC = 786
	B = 57	AC = 209		B = 42	AC = 137	
	C = 49	BC = 170		C = 28	BC = 116	
18	A = 72	AB = 248	ABC = 1559	A = 72	AB = 189	ABC = 871
	B = 64	AC = 243		B = 45	AC = 134	
	C = 57	BC = 219		C = 34	BC = 141	

8.0 Table 3: Summary of Minimum Aberration Projection (MAP): Frequency distribution of K_k-Value of factor projection in $OA(n,2^1 3^1 4^1)$

	Frequency distribution			
Runs	K – Factor Projection (K-Value)	Distance Function (d_1)	Optimization (d_2)	Decision
6	F_1: (10, 6, 3)	(1,1,1)	(0,1, 2)	d_2
	F_2: (28, 19, 15, 13, 8)	(1,1,0,1,0)	(0,0,1,1,1)	
	F_3: (85, 54)	(1,0)	(0,1)	
7	F_1: (15,10,9,6,6,3)	(1,1,0,1,0)	(0,0,1,1,1)	d_2
	F_2: (45,33,28,23,18,13)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (199,101)	(1,0)	(0,1)	
8	F_1: (21,15,12,10,6,4)	(1,1,0,1,0,0)	(0,0,1,0,1,1)	d_2
	F_2: (66,51,45,29,24,17)	(1,1,1,0,0)	(0,0,1,1,1)	
	F_3: (316,111)	(1,0)	(0,1)	
9	F_1: (28,21,16,15,9,7)	(1,1,0,1,0,0)	(0,0,1,0,1,1)	d_2
	F_2: (181,73,61,39,33,30)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (460,176)	(1,0)	(0,1)	
10	F_1: (36,28,21,20,12,8)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	d_2
	F_2: (120,87,79,52,40,30)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (523,165)	(1,0)	(0,1)	
11	F_1: (37,36,28,25,17,10)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	d_2
	F_2: (129,106, 87,70, 55,39)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (580,268)	(1,0)	(0,1)	
12	F_1: (39,37,36,30, 18,12)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	d_2
	F_2: (145,119,80,66,46)	(1,2,0,0,0)	(0,0,1,1,1)	
	F_3: (730,312)	(1,0)	(0,1)	
13	F_1: (42,39,37,36,23,15)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	d_2
	F_2: (151,126,123,97,83,71)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (694, 398)	()	()	
14	F_1: (46,42,39, 26,20)	(1,1,1,0,0)	(0,1,0,1,1)	d_2
	F_2: (156, 131,119, 103,	(1,1,1,0,0,0)	(0,0,0,1,1,1)	

	102,78)			
	F_3: (919, 649)	(0,1)	(1,0)	
15	F_1: (51,49,46,42,31,25)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	d_2
	F_2: (173,137,132,129,113,82)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (739, 663)	(1,0)	(0,1)	
16	F_1: (57,56,47,39,37)	(1,0,2,0,0)	(0,1,0,1,1)	d_2
	F_2: (172,169,159,158,153,126)	(0,1,1,1,0,0)	(1,0,0,0,1,1)	
	F_3: (846,786)	(1,0)	(0,1)	
17	F_1: (64,57,49,42,28)	(1,1,1,0,0)	(1,0,0,1,1)	d_2
	F_2: (238,209,180,170,137,116)	(1,1,0,1,0,0)	(0,0,1,0,1,1)	
	F_3: (1262, 786)	(1,0)	(0,1)	
18	F_1: (72,64,57,45,34)	(1,1,1,0,0)	(1,0,0,1,1)	d_2
	F_2: (248,243,219,189,141,134)	(1,1,1,0,0,0)	(0,0,0,1,1,1)	
	F_3: (1559, 871)	(1,0)	(0,1)	

9.0 Table 4: Summary of design comparison

Balance coefficient form		$OA(n,2^1 3^1 4^1)$	
	Runs	**MMAC**	**MAP**
Optimization	**6**	d_2	d_2
Optimization	**7**	d_2	d_2
Optimization	**8**	d_2	d_2
Optimization	**9**	d_2	d_2
Optimization	**10**	d_2	d_2
Optimization	**11**	d_2	d_2
Optimization	**12**	d_2	d_2
Optimization	**13**	d_2	d_2
Optimization	**14**	d_2	d_2
Optimization	**15**	d_2	d_2
Optimization	**16**	d_2	d_2
Optimization	**17**	d_2	d_2
Optimization	**18**	d_2	d_2

10.0 CONCLUSION

The result indicated that, the MMA and MAP criteria was efficient using the optimization procedure of the balance coefficient.

REFERENCES

[1] Cheng, C. S., Steinberg, D. M., & Sun L. X. (1999). Minimum aberration and model robustness for two-level fractional factorial design. J. R. Stat. Soc. Ser. B Stat. Methodol. 61 85–93.

[2] Chen, H. and Cheng, C.-S. (1999). Theory of optimal blocking of 2^{n-m} designs. Ann. Statist. 27 1948–1973.

[3] Chen, H. and Hedayat, A. S. (1996). 2^{n-m} designs with weak minimum aberration. Ann. Statist. 24 2536–2548.

[4] Chen, J. (1992). Some results on 2^{n-k} fractional factorial designs and search for minimum aberration designs. Ann. Statist. 20 2124–2141.

[5] Chen, J. and Wu, C. F. J. (1991). Some results on sn−k fractional factorial designs with minimum aberration or optimal moments. Ann. Statist. 19 1028–1041.

[6] Cheng, S.-W. and Wu, C. F. J. (2002). Choice of optimal blocking schemes in two-level and three-level designs. Technometrics 44 269–277.

[7] Cheng, S.W. and Ye, K.Q. (2004): Geometric isomorphism and minimum aberration for factorial designs with quantitative factors. Ann. Statist., 32, 2168–2185.

[8] Deng, L.Y. & Tang, B. (2002). Design selection and classification for Hadamard matrices using generalized minimum aberration criteria. Technometrics, 44, 173-184.

[9] Fries, A. & Hunter, W.G. (1980). Minimum aberration 2^{k-p} designs. Technometrics, 22, 601-608.

[10] Franklin, M. F. (1984). Constructing tables of minimum aberration p^{n-m} designs. Technometrics 26 225–232.

[11] Mukerjee, R. & Wu, C. F. J. (2001). A Modern Theory of Factional Designs. New York: Springer.

[12] Sitter, R. R., Chen, J., and Feder, M. (1997). Fractional resolution and minimum aberration in blocked 2n−k designs. Technometrics 39 382–390.

[13] Suen, C.-Y., Chen, H., and Wu, C. F. J. (1997). Some identities on q^{n-m} designs with application to minimum aberration designs. Ann. Statist. 25 1176–1188.

[14] Tang, B. and Deng, L. (1999). Minimum G2-aberration for nonregular fractional factorial designs. Ann. Statist. 27 1914–1926.

[15] Tang, B. and Wu, C. F. J. (1996). Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs. Ann. Statist. 24 2549–2559.

[16] Wang, J.C., and Wu, C.F.J. (1992), “Nearly Orthogonal Designs with Mixed Levels and Small Runs” Technometrices, 34, 409-422

[17] Wu, C. F. J. and Hamada, M. (2000). Experiments: Plan-

ISSN 2229-5518

ning, Analysis, and Parameter Design Optimization. Wiley, New York.

[18] Xu, H. (2002), "An Algorithm for Construction Orthogonal and Nearly-Orthogonal Designs with Mixed Levels and Small Runs" Technometrices, 44, 356-368.

[19] Xu, H. and Wu, C.F.J. (2001): Generalized minimum aberration for asymmetrical fractional factorial designs. Ann. Statist., 29, 1066–1077.

[20] Xu, H. and Deng, L.Y. (2005): Moment aberration projection for nonregular fractional factorial designs. Technometrics, 47, 121–131.

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